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Chapter 3

Some Types of HyperNeutrosophic Set: Bipolar, Pythagorean, Double-Valued, Interval-Valued Set

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Abstract

The Neutrosophic Set is a mathematical framework designed to manage uncertainty, characterized by three membership functions: truth (T), indeterminacy (I), and falsity (F). In recent years, extensions such as the Hyperneutrosophic Set and SuperHyperneutrosophic Set have been introduced to address more complex scenarios. This paper proposes new concepts by extending Bipolar Neutrosophic Sets, Interval-Valued Neutrosophic Sets, Pythagorean Neutrosophic Sets, and Double-Valued Neutrosophic Sets using the frameworks of Hyperneutrosophic and SuperHyperneutrosophic Sets. Additionally, a brief analysis of these extended concepts is presented.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section outlines the essential concepts and definitions required for the discussions in this paper. For a more comprehensive understanding of foundational set theory, readers may consult references such as [24,40,42,45].

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

To better address uncertainty and imprecision in decision-making, several set-theoretic models have been developed, including Fuzzy Sets [69–73], Neutrosophic Sets [26, 32–35, 37, 57, 58, 62], Plithogenic Sets [25,27,28,36,60,61,63], and Soft Sets [48,51].

Neutrosophic Sets extend Fuzzy Sets by introducing the concept of indeterminacy alongside truth and falsity [55–58]. This idea has been further developed into HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets to handle even more complex scenarios [25,29]. The following section provides their succinct definitions and relevant information.

Definition 1.1 (Neutrosophic Set)**.** [57, 58] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 1.2 (Neutrosophic Set in Real Life: Medical Diagnosis)**.** (cf. [18,66])
Consider $X = \{\text{Patient A, Patient B, Patient C}\}$, the set of patients in a hospital. A Neutrosophic Set A is used to evaluate the presence of a disease D for each patient, where:

- $T_A(x)$ represents the degree of truth that the patient has the disease based on test results.
- $I_A(x)$ represents the degree of indeterminacy, accounting for inconclusive test results or lack of information.
- $F_A(x)$ represents the degree of falsity that the patient has the disease.

For example:

$$T_A(\text{Patient A}) = 0.8, \quad I_A(\text{Patient A}) = 0.1, \quad F_A(\text{Patient A}) = 0.1,$$

indicating that there is a high likelihood (80%) that Patient A has the disease, with minimal uncertainty (10%) and falsity (10%).

Definition 1.3 (HyperNeutrosophic Set)**.** (cf. [25,29–31,59]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \le T + I + F \le 3.$$

Example 1.4 (HyperNeutrosophic Set in Real Life: Restaurant Review Analysis)**.** Consider

$$X = \{\text{Restaurant X, Restaurant Y, Restaurant Z}\}$$

, the set of restaurants. A HyperNeutrosophic Set $\tilde{A}$ maps each restaurant to subsets of $[0,1]^3$, where:

- (T, I, F) represents customer feedback in terms of truth (T) for positive reviews, indeterminacy (I) for neutral or unclear reviews, and falsity (F) for negative reviews.
- Multiple triplets can represent diverse opinions.

For example:

$$\tilde{\mu}(\text{Restaurant X}) = \{(0.9, 0.05, 0.05), (0.7, 0.2, 0.1)\},$$

indicating most customers rate it positively with slight variation in indeterminacy and falsity. Another restaurant:

$$\tilde{\mu}(\text{Restaurant Y}) = \{(0.4, 0.4, 0.2), (0.6, 0.3, 0.1)\},$$

shows mixed feedback with higher uncertainty in reviews.

Definition 1.5 (n-SuperHyperNeutrosophic Set)**.** (cf. [25, 29–31, 59]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \ge 2$,

  $$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \le T + I + F \le 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Bipolar Hyperneutrosophic set

A Bipolar Neutrosophic Set (BNS) represents elements with positive and negative truth, indeterminacy, and falsity membership functions, handling dual perspectives [1, 6–9, 11, 13, 22, 50, 54, 65]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.1 (Bipolar Neutrosophic Set)**.** (cf. [7, 11, 54]) Let X be a non-empty set. A *Bipolar Neutrosophic Set (BNS) A* in X is defined as:

$$A = \left\{ \langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x) \rangle : x \in X \right\},$$

where:

- $T^+, I^+, F^+ : X \to [0, 1]$ are the positive truth-membership, indeterminacy-membership, and falsity-membership functions, respectively.
- $T^-, I^-, F^- : X \to [-1, 0]$ are the negative truth-membership, indeterminacy-membership, and falsity-membership functions, respectively.

Here:

- $T^+(x), I^+(x), F^+(x)$ represent the degrees of truth, indeterminacy, and falsity for an element x in relation to a positive property.
- $T^-(x), I^-(x), F^-(x)$ represent the degrees of truth, indeterminacy, and falsity for an element x in relation to an implicit counter-property.

Definition 2.2 (Bipolar Hyperneutrosophic Set (BHNS))**.** Let X be a non-empty set. A *Bipolar Hyperneutrosophic Set $\widetilde{B}$* on X is a mapping

$$\widetilde{B} : X \to \mathcal{P}\Big([0, 1]^3 \times [-1, 0]^3\Big),$$

such that for every $x \in X$, $\widetilde{B}(x)$ is a non-empty subset of $[0, 1]^3 \times [-1, 0]^3$ whose generic element can be written as $\big((T^+, I^+, F^+), (T^-, I^-, F^-)\big)$, subject to:

$$\begin{aligned} 0 \ &\le \ T^+ + I^+ + F^+ \ \le \ 3, \\ -3 \ &\le \ T^- + I^- + F^- \ \le \ 0. \end{aligned}$$

Here:

- $(T^+, I^+, F^+) \in [0, 1]^3$ quantifies the positive truth, indeterminacy, and falsity for x,
- $(T^-, I^-, F^-) \in [-1, 0]^3$ quantifies the negative truth, indeterminacy, and falsity for x,
- each $x \in X$ may have multiple such pairs in $\widetilde{B}(x)$, reflecting a *set-valued* or *hyper* perspective of bipolar neutrosophic membership.

Theorem 2.3. *Every Bipolar Neutrosophic Set is a special case of a Bipolar Hyperneutrosophic Set.*

Proof. A Bipolar Neutrosophic Set A on X associates each $x \in X$ with exactly one 6-tuple

$$\big(T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\big)$$

. We can embed this in Definition 2.2 by letting

$$\widetilde{B}(x) \ = \ \Big\{\big(\big(T^+(x), I^+(x), F^+(x)\big), \big(T^-(x), I^-(x), F^-(x)\big)\big)\Big\} \subseteq [0, 1]^3 \times [-1, 0]^3.$$

Hence, each x maps to a *singleton set* containing the same 6-tuple from the BNS context. The constraints on $T^+ + I^+ + F^+$ and $T^- + I^- + F^-$ remain unchanged. Consequently, every BNS is realized as a special (single-valued) case of a BHNS. □

Theorem 2.4. *Every Hyperneutrosophic Set can be regarded as a special case of a Bipolar Hyperneutrosophic Set by nullifying its "negative" side.*

Proof. A Hyperneutrosophic Set $\tilde{A}$ has $\tilde{A}(x) \subseteq [0,1]^3$ with the condition $0 \le T + I + F \le 3$. In a BHNS, each $\widetilde{B}(x)$ is a subset of $[0,1]^3 \times [-1,0]^3$. If we force each (T^-, I^-, F^-) to be identically $(0,0,0)$, we essentially collapse the negative dimension. Define

$$\widetilde{B}(x) = \Big\{ \big((T, I, F), (0,0,0)\big) : (T, I, F) \in \tilde{A}(x) \Big\}.$$

Thus, $\widetilde{B}(x)$ only varies in the first (positive) triplet, effectively matching the Hyperneutrosophic membership. All conditions remain consistent, and no negativity is introduced. This recovers the exact structure of an HNS as a special BHNS case. □

Definition 2.5 (Bipolar n-SuperHyperneutrosophic Set (B-n-SHNS))**.** Let X be a non-empty set, and consider the nested power sets $\mathcal{P}_n(X)$ defined by

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \text{ for } k \ge 2.$$

Similarly, let

$$\mathcal{P}_n\Big([0,1]^3 \times [-1,0]^3\Big)$$

denote the n-nested family of non-empty subsets of the product space $[0,1]^3 \times [-1,0]^3$.

A *Bipolar n-SuperHyperneutrosophic Set* is a mapping

$$\widetilde{B}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\Big([0,1]^3 \times [-1,0]^3\Big),$$

such that for any $A \in \mathcal{P}_n(X)$, $\widetilde{B}_n(A)$ is a (non-empty) subset of $[0,1]^3 \times [-1,0]^3$-valued "degrees of bipolar neutrosophic membership" satisfying the constraints:

$$\begin{aligned} 0 \ &\le \ T^+ + I^+ + F^+ \ \le \ 3, \\ -3 \ &\le \ T^- + I^- + F^- \ \le \ 0, \end{aligned} \qquad \text{for each } \big((T^+, I^+, F^+), (T^-, I^-, F^-)\big) \in \widetilde{B}_n(A).$$

In other words, each A at the n-th nesting level is assigned a set of 6-tuples combining positive and negative membership, and each 6-tuple is bounded by the usual neutrosophic constraints of total membership in $[0,3]$ for positivity and $[-3,0]$ for negativity.

Theorem 2.6. *Every Bipolar Hyperneutrosophic Set is a particular case of a Bipolar n-SuperHyperneutrosophic Set (B-n-SHNS).*

Proof. A Bipolar Hyperneutrosophic Set $\widetilde{B}$, as defined in Definition 2.2, deals with elements $x \in X$ (so basically $n = 1$). In a B-n-SHNS from Definition 2.5, let $n = 1$, so $\mathcal{P}_1(X) = \mathcal{P}(X)$, but we only ever evaluate $\widetilde{B}_n(\{x\})$ for singletons $\{x\} \subseteq X$. Define

$$\widetilde{B}_1\big(\{x\}\big) := \widetilde{B}(x),$$

and $\widetilde{B}_1(A) := \varnothing$ (or some consistent assignment) for any $A \subseteq X$ with $|A| \neq 1$. Under this construction, we preserve all bipolarly hyperneutrosophic membership values from $\widetilde{B}$. Hence, the B-n-SHNS with $n = 1$ exactly replicates the BHNS membership in the special case where $A = \{x\}$. Therefore, any BHNS is embedded in a B-1-SHNS as a restricted scenario. □

Theorem 2.7. *Every n-SuperHyperneutrosophic Set is a special case of a Bipolar n-SuperHyperneutrosophic Set, obtained by nullifying negative membership.*

Proof. An n-SuperHyperneutrosophic Set [62] is a mapping

$$\tilde{A}_n : \mathcal{P}_n(X) \ \to \ \mathcal{P}_n([0,1]^3),$$

satisfying $0 \leq T+I+F \leq 3$ for each $(T, I, F) \in \tilde{A}_n(A)$. In the Bipolar n-SuperHyperneutrosophic Set context, each $\widetilde{B}_n(A)$ is a subset of $([0,1]^3 \times [-1,0]^3)$. We can force the negative part to be $(0,0,0)$, similarly to Theorem 2.4. Concretely, define

$$\widetilde{B}_n(A) = \Big\{\big((T, I, F), (0,0,0)\big) : (T, I, F) \in \tilde{A}_n(A)\Big\}.$$

All constraints remain satisfied: $0 \leq T+I+F \leq 3$ is preserved, and $T^- + I^- + F^- = 0$ lies in $[-3, 0]$. Thus, each n-SuperHyperneutrosophic membership is recovered from a Bipolar n-SuperHyperneutrosophic membership by ignoring negativity. Consequently, we obtain an n-SuperHyperneutrosophic Set as a special case of B-n-SHNS by nullifying the negative portion. □

2.2 Pythagorean Neutrosophic Set

A Pythagorean Neutrosophic Set defines truth, indeterminacy, and falsity degrees for elements, satisfying a squared-sum constraint [2–5, 14, 19, 20, 41, 52, 53]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.8 (Pythagorean Neutrosophic Set)**.** (cf. [2, 3, 19]) Let X be a non-empty set (universe). A *Pythagorean Neutrosophic Set (PNS) A* on X is defined as:

$$A = \{\langle x, u_A(x), \zeta_A(x), \nu_A(x)\rangle : x \in X\},$$

where:

- $u_A(x), \zeta_A(x), \nu_A(x) \in [0,1]$ for all $x \in X$,
- $u_A(x), \nu_A(x)$ are dependent components (membership and non-membership degrees),
- $\zeta_A(x)$ is an independent component (indeterminacy degree), and
- the following condition holds:

$$0 \leq (u_A(x))^2 + (\zeta_A(x))^2 + (\nu_A(x))^2 \leq 2, \quad \forall x \in X.$$

Definition 2.9 (Pythagorean Hyperneutrosophic Set (PHNS))**.** Let X be a non-empty set. A *Pythagorean Hyperneutrosophic Set (PHNS) $\widetilde{A}$* on X is a mapping

$$\widetilde{A} : X \rightarrow \mathcal{P}\big([0,1]^3\big),$$

such that for each $x \in X$, the image $\widetilde{A}(x)$ is a non-empty subset of $[0,1]^3$ whose generic element is a triplet (T, I, F) satisfying both

$$0 \leq T+I+F \leq 3 \quad \text{(the usual neutrosophic/hyperneutrosophic constraint)},$$

and the *Pythagorean* constraint

$$(T)^2 + (I)^2 + (F)^2 \leq 2.$$

Hence, each $x \in X$ is associated with multiple Pythagorean neutrosophic membership triplets in a *set-valued* manner.

Theorem 2.10 (PHNS Generalizes PNS)**.** *Every Pythagorean Neutrosophic Set is a special case of a Pythagorean Hyperneutrosophic Set.*

Proof. A *Pythagorean Neutrosophic Set (PNS)* on X is given by

$$A = \Big\{\langle x, u_A(x), \zeta_A(x), \nu_A(x)\rangle : x \in X\Big\},$$

with each $(u_A(x))^2 + (\zeta_A(x))^2 + (\nu_A(x))^2 \leq 2$ and all components in $[0,1]$. To embed this into Definition 2.9, define a mapping $\widetilde{A}$ by:

$$\widetilde{A}(x) = \Big\{\big(u_A(x),\ \zeta_A(x),\ \nu_A(x)\big)\Big\} \subseteq [0,1]^3.$$

Hence, for each $x \in X$, $\widetilde{A}(x)$ is a *singleton set* containing exactly one triplet. Clearly, $u_A(x)+\zeta_A(x)+\nu_A(x) \le 3$ and $u_A(x)^2+\zeta_A(x)^2+\nu_A(x)^2 \le 2$ are satisfied by assumption. Therefore, each single triplet meets the required conditions:

$$0 \le u_A(x) + \zeta_A(x) + \nu_A(x) \le 3, \quad (u_A(x))^2 + (\zeta_A(x))^2 + (\nu_A(x))^2 \le 2.$$

Thus, $(X, \widetilde{A})$ is a Pythagorean Hyperneutrosophic Set that *coincides* with the given PNS in a single-valued manner. This shows every PNS is a special (singleton-valued) case of a PHNS. □

Theorem 2.11 (PHNS Generalizes HNS)**.** *Every Hyperneutrosophic Set is a special case of a Pythagorean Hyperneutrosophic Set by dropping the Pythagorean constraint.*

Proof. A *Hyperneutrosophic Set (HNS)* $\tilde{\mu}$ satisfies

$$\tilde{\mu}(x) \subseteq \{(T, I, F) \in [0,1]^3 : 0 \le T + I + F \le 3\}.$$

In Definition 2.9, we have the additional constraint $(T)^2 + (I)^2 + (F)^2 \le 2$. If we *omit* or do not enforce $(T)^2 + (I)^2 + (F)^2 \le 2$, we recover a standard HNS structure: let

$$\widetilde{A}(x) = \tilde{\mu}(x) \quad \text{for all } x \in X,$$

and ignore the Pythagorean condition. This matches exactly the hyperneutrosophic membership sets in $[0,1]^3$ with $T + I + F \le 3$, thus reproducing an HNS. Hence the PHNS concept, with the Pythagorean constraint relaxed, coincides with a standard HNS. This shows HNS is strictly contained within PHNS if the Pythagorean constraint is optional. □

Definition 2.12 (Pythagorean n-SuperHyperneutrosophic Set (P-n-SHNS))**.** Let X be a non-empty set, and recall the recursively defined families:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \text{ for } k \ge 2.$$

Similarly define $\mathcal{P}_n([0,1]^3)$ for the nested power sets of $[0,1]^3$.

A *Pythagorean n-SuperHyperneutrosophic Set (P-n-SHNS)* is a mapping

$$\widetilde{B}_n : \mathcal{P}_n(X) \;\to\; \mathcal{P}_n\big([0,1]^3\big),$$

such that for any $A \in \mathcal{P}_n(X)$, each triplet $(T, I, F) \in \widetilde{B}_n(A)$ satisfies:

$$\begin{aligned} 0 \;&\le\; T + I + F \;\le\; 3, \\ (T)^2 + (I)^2 + (F)^2 \;&\le\; 2. \end{aligned}$$

In other words, each n-th level subset A is assigned a set of Pythagorean membership triplets in $[0,1]^3$, each fulfilling the usual neutrosophic/hyperneutrosophic boundary plus the Pythagorean condition.

Theorem 2.13 (P-n-SHNS Generalizes PHNS)**.** *Any Pythagorean Hyperneutrosophic Set is a particular case of a Pythagorean n-SuperHyperneutrosophic Set.*

Proof. A *Pythagorean Hyperneutrosophic Set (PHNS)* $\widetilde{A}$ assigns each $x \in X$ a subset $\widetilde{A}(x) \subseteq [0,1]^3$ of triplets fulfilling $T + I + F \le 3$ and $T^2 + I^2 + F^2 \le 2$. In a P-n-SHNS from Definition 2.12, choose $n = 1$, so $\mathcal{P}_1(X) = \mathcal{P}(X)$. We can define

$$\widetilde{B}_1(\{x\}) := \widetilde{A}(x)$$

and assign $\widetilde{B}_1(A) := \varnothing$ (or some consistent choice) for any $A \subseteq X$ with $|A| \neq 1$. In that case, for singletons $A = \{x\}$, we replicate precisely the membership sets from the PHNS. The constraints $T^2 + I^2 + F^2 \le 2$ and $T + I + F \le 3$ remain the same. Hence, $\widetilde{B}_1$ is exactly the given PHNS in restricted form. This shows any PHNS is realized as a special ($n = 1$) instance of a P-n-SHNS. □

Theorem 2.14 (P-n-SHNS Generalizes n-SHNS)**.** *Every n-SuperHyperneutrosophic Set is a special case of a Pythagorean n-SuperHyperneutrosophic Set if we discard the Pythagorean constraint.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ satisfies $T + I + F \leq 3$ for all triplets $(T, I, F) \in \tilde{A}_n(A)$, where $A \in \mathcal{P}_n(X)$. Compare this with Definition 2.12, which adds $(T)^2 + (I)^2 + (F)^2 \leq 2$. If we simply do *not* enforce the Pythagorean condition, we reproduce the original n-SHNS constraints. Formally, for a given $\tilde{A}_n$, define

$$\widetilde{B}_n(A) := \tilde{A}_n(A) \quad \text{for all } A \in \mathcal{P}_n(X),$$

and do *not* impose $(T)^2 + (I)^2 + (F)^2 \leq 2$. Then $\widetilde{B}_n$ matches exactly the membership sets from $\tilde{A}_n$. Therefore, ignoring Pythagorean constraints yields an ordinary n-SHNS. Consequently, any n-SHNS is a special case of a P-n-SHNS in which we disregard the extra Pythagorean condition. □

2.3 Double-Valued Neutrosophic Set

A Double-Valued Neutrosophic Set represents truth, indeterminacy (toward truth/falsity), and falsity degrees for elements, summing up to ≤ 4 [23,38,39,43,44,46,47,67,76,77]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.15 (Double-Valued Neutrosophic Set)**.** [43] Let X be a non-empty set (universe). A *Double-Valued Neutrosophic Set (DVNS) A* on X is defined as:

$$A = \{\langle x, T_A(x), I_T(x), I_F(x), F_A(x)\rangle : x \in X\},$$

where:

- $T_A(x), I_T(x), I_F(x), F_A(x) \in [0, 1]$ for all $x \in X$,
- $T_A(x)$: truth membership degree,
- $I_T(x)$: indeterminacy leaning towards truth,
- $I_F(x)$: indeterminacy leaning towards falsity,
- $F_A(x)$: falsity membership degree,
- the following condition holds:

$$0 \leq T_A(x) + I_T(x) + I_F(x) + F_A(x) \leq 4, \quad \forall x \in X.$$

Definition 2.16 (Double-Valued Hyperneutrosophic Set (DVHNS))**.** Let X be a non-empty set. A *Double-Valued Hyperneutrosophic Set $\widetilde{D}$* on X is a mapping

$$\widetilde{D} : X \rightarrow \mathcal{P}\big([0,1]^4\big),$$

where $\mathcal{P}([0,1]^4)$ denotes the family of all non-empty subsets of $[0,1]^4$. For each $x \in X$, the set $\widetilde{D}(x) \subseteq [0,1]^4$ consists of quadruples $\big(T, I_T, I_F, F\big)$ that satisfy

$$0 \leq T + I_T + I_F + F \leq 4.$$

Here:

- T = truth-membership degree,
- I_T = indeterminacy leaning towards truth,
- I_F = indeterminacy leaning towards falsity,
- F = falsity-membership degree.

Each $x \in X$ may be associated with *multiple* such quadruples, forming a *set-valued* membership structure.

Theorem 2.17. *Every Double-Valued Neutrosophic Set is a special case of a Double-Valued Hyperneutrosophic Set.*

Proof. A *Double-Valued Neutrosophic Set (DVNS) A* on X assigns each $x \in X$ a unique 4-tuple

$$\big(T_A(x), I_T(x), I_F(x), F_A(x)\big)$$

with $T_A + I_T + I_F + F_A \leq 4$. We embed this into Definition 2.16 by letting

$$\widetilde{D}(x) = \Big\{\big(T_A(x),\ I_T(x),\ I_F(x),\ F_A(x)\big)\Big\} \ \subseteq\ [0,1]^4.$$

Hence, each x is mapped to a *singleton set* containing precisely the same quadruple. The condition $T_A + I_T + I_F + F_A \leq 4$ remains unchanged. Therefore, each single-valued DVNS is captured as a special (singleton) DVHNS. □

Theorem 2.18. *Every Hyperneutrosophic Set is a special case of a Double-Valued Hyperneutrosophic Set when the extra (fourth) component is nullified.*

Proof. A *Hyperneutrosophic Set (HNS)* $\tilde{\mu}$ satisfies

$$\tilde{\mu}(x) \subseteq \Big\{(T, I, F) \in [0,1]^3 : 0 \leq T + I + F \leq 3\Big\}.$$

In Definition 2.16, each membership is a subset of $[0,1]^4$ with $T + I_T + I_F + F \leq 4$. We reduce to HNS by forcing $I_F = 0$. Concretely, define

$$\widetilde{D}(x) = \Big\{\big(T, I, 0, F\big) \ \mid\ (T, I, F) \in \tilde{\mu}(x)\Big\} \ \subseteq\ [0,1]^4.$$

Then the condition $T + I + 0 + F \leq 4$ is effectively $T + I + F \leq 4$. By restricting further to $T + I + F \leq 3$ (which is typically satisfied in HNS), we see that ignoring the extra dimension recovers the standard 3-component condition. Thus, HNS is included within DVHNS by identifying the extra dimension with zero. □

Definition 2.19 (Double-Valued n-SuperHyperneutrosophic Set (DV-n-SHNS))**.** Let X be a non-empty set, and let $\mathcal{P}_n(X)$ be the n-th nested power set of X, defined by:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \text{ for } k \geq 2.$$

Similarly, define $\mathcal{P}_n([0,1]^4)$ for the nested power sets of $[0,1]^4$.

A *Double-Valued n-SuperHyperneutrosophic Set* $\widetilde{D}_n$ is a mapping:

$$\widetilde{D}_n \ :\ \mathcal{P}_n(X) \ \longrightarrow\ \mathcal{P}_n\big([0,1]^4\big),$$

such that for any $A \in \mathcal{P}_n(X)$ and for any quadruple $\big(T, I_T, I_F, F\big) \in \widetilde{D}_n(A) \subseteq [0,1]^4$, the following holds:

$$0 \ \leq\ T + I_T + I_F + F \ \leq\ 4.$$

Theorem 2.20. *Every Double-Valued Hyperneutrosophic Set is a particular case of a Double-Valued n-SuperHyperneutrosophic Set.*

Proof. A *Double-Valued Hyperneutrosophic Set (DVHNS)* $\widetilde{D}$ has $\widetilde{D}(x) \subseteq [0,1]^4$ for each $x \in X$. In a DV-n-SHNS (Definition 2.19), choose $n = 1$. Then

$$\widetilde{D}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \ \rightarrow\ \mathcal{P}_1\big([0,1]^4\big) = \mathcal{P}\big([0,1]^4\big).$$

We can define:

$$\widetilde{D}_1(\{x\}) = \widetilde{D}(x), \quad \text{and possibly set } \widetilde{D}_1(A) = \varnothing \text{ for other } A \neq \{x\}.$$

Hence, restricting to singletons $\{x\} \subseteq X$ recovers exactly the DVHNS. Thus, the DVHNS is embedded in the DV-1-SHNS as a special case. □

Theorem 2.21. *Every n-SuperHyperneutrosophic Set is a special case of a Double-Valued n-SuperHyperneutrosophic Set by nullifying the extra dimension.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ maps $A \in \mathcal{P}_n(X)$ to subsets of $[0,1]^3$, each satisfying $T + I + F \leq 3$. In DV-n-SHNS, each membership is in $[0,1]^4$ with $T + I_T + I_F + F \leq 4$. To match an SHNS, we can do the following for each A:

$$\widetilde{D}_n(A) = \Big\{\big(T, I, 0, F\big) : (T, I, F) \in \tilde{A}_n(A)\Big\}.$$

We also may require $T + I + F \leq 3$ to remain consistent, embedded in $T + I + (0) + F \leq 4$. This effectively sets $I_F = 0$, reducing the dimension to 3. Thus, ignoring the fourth component recovers the usual n-SHNS form. Therefore, every n-SHNS is embedded in DV-n-SHNS by trivializing the extra dimension. □

2.4 Interval-Valued Neutrosophic Set

An Interval-Valued Neutrosophic Set assigns interval-based truth, indeterminacy, and falsity degrees to elements, capturing uncertainty within specified ranges [10, 12, 15–17, 21, 49, 64, 68, 74, 75]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.22 (Interval-Valued Neutrosophic Set)**.** (cf. [21, 68, 74, 75]) Let X be a non-empty set (universe). An *Interval-Valued Neutrosophic Set (IVNS) A* on X is defined as:

$$A = \left\{\langle x, [T_A^l(x), T_A^r(x)], [I_A^l(x), I_A^r(x)], [F_A^l(x), F_A^r(x)]\rangle : x \in X\right\},$$

where:

- $[T_A^l(x), T_A^r(x)]$: interval of truth membership degrees,
- $[I_A^l(x), I_A^r(x)]$: interval of indeterminacy membership degrees,
- $[F_A^l(x), F_A^r(x)]$: interval of falsity membership degrees,
- $T_A^l(x), T_A^r(x), I_A^l(x), I_A^r(x), F_A^l(x), F_A^r(x) \in [0,1]$,
- and the condition:
$$0 \leq T_A^r(x) + I_A^r(x) + F_A^r(x) \leq 3, \quad \forall x \in X.$$

Definition 2.23 (Interval-Valued Hyperneutrosophic Set (IVHNS))**.** Let X be a non-empty set. An *Interval-Valued Hyperneutrosophic Set (IVHNS) $\widetilde{H}$* on X is a mapping

$$\widetilde{H} : X \;\to\; \mathcal{P}\big([0,1]^6\big),$$

where each $\widetilde{H}(x) \subseteq [0,1]^6$ is a (non-empty) set of *interval-triplets*

$$\Big(\big(T^l, T^r\big), \big(I^l, I^r\big), \big(F^l, F^r\big)\Big),$$

subject to:

1. $0 \leq T^l \leq T^r \leq 1, \quad 0 \leq I^l \leq I^r \leq 1, \quad 0 \leq F^l \leq F^r \leq 1,$
2. The *upper bounds* satisfy:
$$T^r + I^r + F^r \quad \leq \; 3.$$

In other words, for each $x \in X$, $\widetilde{H}(x)$ is a set of intervals describing the truth, indeterminacy, and falsity degrees in $[0,1]$ such that the sum of the *right endpoints* does not exceed 3.

Theorem 2.24 (IVHNS Generalizes IVNS)**.** *Every Interval-Valued Neutrosophic Set is a particular case of an Interval-Valued Hyperneutrosophic Set.*

Proof. An *Interval-Valued Neutrosophic Set (IVNS) A* on X is given by

$$A = \Big\{\langle x, [T_A^l(x), T_A^r(x)], [I_A^l(x), I_A^r(x)], [F_A^l(x), F_A^r(x)]\rangle \;:\; x \in X\Big\},$$

where $T_A^r(x) + I_A^r(x) + F_A^r(x) \le 3$. To embed this in Definition 2.23, define a set-valued mapping $\widetilde{H}$ by:

$$\widetilde{H}(x) = \Big\{\big((T_A^l(x),\, T_A^r(x)), (I_A^l(x),\, I_A^r(x)), (F_A^l(x),\, F_A^r(x))\big)\Big\} \subseteq [0,1]^6.$$

Hence, each x maps to a *singleton set* containing exactly one interval-triplet. The condition on the right endpoints ≤ 3 is the same. Therefore, each single-valued IVNS is captured as a special (singleton) IVHNS. □

Theorem 2.25 (IVHNS Generalizes HNS)**.** *Every Hyperneutrosophic Set is a special case of an Interval-Valued Hyperneutrosophic Set by restricting intervals to single points.*

Proof. A *Hyperneutrosophic Set (HNS)* $\tilde{\mu}$ assigns each $x \in X$ a subset of $[0,1]^3$, with (T, I, F) satisfying $T + I + F \le 3$. In the IVHNS of Definition 2.23, each membership is a subset of $[0,1]^6$ of interval-triplets $\big(T^l, T^r, I^l, I^r, F^l, F^r\big)$. If we *force* each pair (T^l, T^r) to collapse to (T, T), (I^l, I^r) to (I, I), and (F^l, F^r) to (F, F), then effectively

$$\widetilde{H}(x) = \Big\{\big((T,T), (I,I), (F,F)\big) : (T, I, F) \in \tilde{\mu}(x)\Big\}.$$

The sum-of-right-endpoints constraint becomes $T + I + F \le 3$. This matches the standard HNS membership. Hence, HNS emerges as a special case of IVHNS by identifying intervals with their single-point degenerate intervals. □

Definition 2.26 (Interval-Valued n-SuperHyperneutrosophic Set (IV-n-SHNS))**.** Let X be a non-empty set, and define

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \;\text{ for } k \ge 2.$$

Similarly, we define $\mathcal{P}_n([0,1]^6)$ for the nested power set of $[0,1]^6$. An *Interval-Valued n-SuperHyperneutrosophic Set (IV-n-SHNS)* is a mapping

$$\widetilde{H}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\big([0,1]^6\big),$$

such that for any $A \in \mathcal{P}_n(X)$ and any $\big((T^l, T^r), (I^l, I^r), (F^l, F^r)\big) \in \widetilde{H}_n(A) \subseteq [0,1]^6$, the following hold:

$$0 \le T^l \le T^r \le 1, \quad 0 \le I^l \le I^r \le 1, \quad 0 \le F^l \le F^r \le 1,$$

and

$$T^r + I^r + F^r \le 3.$$

In other words, each n-th level subset $A \subseteq X$ is assigned a *set* of interval-triplets $\big([T^l, T^r], [I^l, I^r], [F^l, F^r]\big)$ satisfying the usual neutrosophic upper-bound constraint on $(T^r + I^r + F^r)$.

Theorem 2.27. *Every Interval-Valued Hyperneutrosophic Set is a special case of an Interval-Valued n-SuperHyperneutrosophic Set.*

Proof. An *Interval-Valued Hyperneutrosophic Set (IVHNS)* $\widetilde{H}$ is a mapping $\widetilde{H} : X \to \mathcal{P}([0,1]^6)$. In IV-$n$-SHNS (Definition 2.26), choose $n = 1$. Then:

$$\widetilde{H}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \;\to\; \mathcal{P}_1\big([0,1]^6\big) = \mathcal{P}\big([0,1]^6\big).$$

Define

$$\widetilde{H}_1(\{x\}) = \widetilde{H}(x), \quad \text{and possibly set } \widetilde{H}_1(A) = \varnothing \text{ for } A \ne \{x\}.$$

Hence, restricting to singletons recovers the IVHNS exactly. Thus, an IVHNS is embedded in IV-1-SHNS as a special case. □

Theorem 2.28. *Every n-SuperHyperneutrosophic Set is a special case of an Interval-Valued n-SuperHyperneutrosophic Set by making each interval degenerate to a point.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ maps $A \in \mathcal{P}_n(X)$ to subsets of $[0,1]^3$, each triplet (T, I, F) with $T + I + F \leq 3$. The IV-n-SHNS in Definition 2.26 uses subsets of $[0,1]^6$ representing intervals $\left(T^l, T^r, I^l, I^r, F^l, F^r\right)$. We can force each interval to collapse to a single point:

$$T^l = T^r = T, \quad I^l = I^r = I, \quad F^l = F^r = F,$$

where $(T, I, F) \in [0,1]^3$. Define

$$\widetilde{H}_n(A) = \left\{\big((T,T),(I,I),(F,F)\big) : (T,I,F) \in \tilde{A}_n(A)\right\}.$$

Hence, $T^r + I^r + F^r = T + I + F \leq 3$ becomes the standard condition. Therefore, ignoring the interval nature yields an n-SHNS. Consequently, any n-SHNS is subsumed under IV-n-SHNS by setting intervals to degenerate points. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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